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Equilibrium and Elasticity

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Overview

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Equilibrium is a special case in mechanics that is very important in everyday life. It occurs when the net force and the net torque on an object or system are both zero. This means that both the linear and angular accelerations are zero. Thus, the object is at rest, or its center of mass is moving at a constant velocity. However, this does not mean that no forces are acting on the objects within the system. In fact, there are very few scenarios on Earth in which no forces are acting upon any given object. If a person walks across a bridge, they exert a downward force on the bridge proportional to their mass, and the bridge exerts an equal and opposite upward force on the person. In some cases, the bridge may flex in response to the downward force of the person, and in extreme cases, when the forces are great enough, the bridge may become seriously deformed or may even fracture. The study of this flexing of objects in equilibrium is called elasticity and becomes extremely important when engineers are designing buildings and structures that we use every day.

Principles

The requirements for a system to obtain equilibrium are simple to write down. In equilibrium, the sum of the forces and the sum of the torques are zero:

$\Sigma \mathbf{F} = 0$ **(Equation 1)**

and

$\Sigma \boldsymbol{\tau} = 0$. **(Equation 2)**

The torque $\boldsymbol{\tau}$ is an angular force, defined as the cross product of the length of the lever arm from where the force is applied to the axis of rotation. That distance is denoted as $\mathbf{r}$:

$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, **(Equation 3)**

$= \mathbf{r}\,\mathbf{F} \sin(\theta)$

where θ is the angle at which the force is applied to the lever arm. For forces perpendicular with respect to the lever arm, **Equation 3** simply becomes $\boldsymbol{\tau} = \mathbf{r} \cdot \mathbf{F}$.

These equations are simple enough to write down, but as the system in question becomes more complex, more forces and torques are involved, and finding the optimum configuration that satisfies equilibrium can become quite difficult. The general approach to solving **Equation 1** is to decompose the forces into the *x*-, *y*- and *z*-directions and then to solve **Equation 1** for each of the three directions (*e.g.*,$\Sigma \mathbf{F}_x = \Sigma \mathbf{F}_y = \Sigma \mathbf{F}_z = 0$). In situations where there is only movement in the *xy*-plane, the torque is calculated about an axis perpendicular to that plane. This axis is arbitrarily chosen to simplify the calculations; if all the objects in the system are at rest, then **Equation 2** will hold true about any axis. In three dimensions, the axis of rotation is again generally chosen such that the calculations are the simplest, which depends upon the configuration of the system. For example, choosing the rotation axis so that one of the unknown forces acts through that axis will result in zero lever arm and produce no torque (see **Equation 3**), making one less term appear in the torque equation. There is no single technique for solving equilibrium problems, but choosing convenient coordinate systems can greatly simplify the process of solving **Equations 1** and **2.**

When the objects in the system undergo equilibrium forces, some of them will compress or expand, depending upon their material and the configuration within the system. For example, when a force is exerted on a rod or spring, its length will expand proportionally to the force, given by Hooke's law:

$\mathbf{F} = k\,\Delta L$, **(Equation 4)**

where ΔL is the length of expansion and k is a constant of proportionality called the "spring constant."

Procedure

1. Observe equilibrium in a static system and verify that the sum of the forces and torques is zero. Confirm the spring constants *k* used in the system.

1. Obtain a meter stick, two spring scales with known spring constants, two stands to suspend the springs from, two weights of different masses, and a mechanism to suspend the weights from the meter stick.
2. Secure the two stands to the table, 1 m apart.
3. Attach the springs to the stands.
4. Attach the spring to each end of the meter stick.

5. Attach the first weight to the middle of the meter stick.
6. Compute both the force and torque exerted by the weight on the meter stick and record them in **Table 1**.
7. Record the force exerted on each of the springs in **Table 1**.
8. Shift the weight to the left by 0.2 m and repeat steps 1.6-1.7.
9. Shift the weight to the left an additional 0.2 m, so the total displacement from the center of the meter stick is 0.4 m. In other words, the length of the moment arm for the spring on the left is 0.1 m, and the length of the moment arm for the spring on the right is 0.9 m.
10. Repeat steps 1.5-1.9 for the other weight.
11. Compute the percent difference of the calculated forces on the left and right springs, $\mathbf{F_L}$ and $\mathbf{F_R}$, against the corresponding forces read off the spring scales.

Results

The representative results for the experiment can be found in **Table 1**. The force exerted on the two springs by the hanging mass are denoted by their locations: left and right, denoted by subscripts **L** and **R.** Since there are two unknowns in this experiment, $\mathbf{F_L}$and $\mathbf{F_R}$, two equations are required to solve for them. Thus, **Equations 1** and **2** are used to solve for the two forces. The torques are used to obtain a relationship between $\mathbf{F_L}$**and** $\mathbf{F_R}$ **.**

Since the force exerted by the weight is downward, the angle θ in **Equation 3** is 90°,and the torque is just $\mathbf{r} \cdot \mathbf{F}$. The torques $\tau_L$and τ_R are also in opposite directions, where counterclockwise is defined as the positive direction. Using **Equation 2**

$-\tau_L + \tau_R = 0 = -r_L F_L + r_R F_R$. **(Equation 5)**

Equivalently,

$F_L = F_R r_R/r_L$. **(Equation 6)**

Using **Equation 1**

$F_L + F_R = m g$, **(Equation 7)**

where **m** is the mass of the weight and *g* is the gravitational constant of 9.8 m/s^2. In other words, the downward force of the weight equals the sum of the forces holding up the weight and meter stick system, which is just the two springs on the left and right, which are suspending the system. With these two equations (**6** and **7**), the unknowns $\mathbf{F_L}$ and $\mathbf{F_R}$ can be calculated. These are shown in **Table 1**. These values are compared with the forces exerted on the springs in the last two columns of the table. Slight discrepancies are expected from measurement errors. In addition, it has been assumed that the mass of the meter stick is zero, which is incorrect, strictly speaking, but nevertheless a good approximation. This lab uses spring scales, which show how many Newtons are being applied to the spring when stretched, so it is not necessary to know the spring constant, *k.*

Table 1. Theoretical and experimental results.

Mass (g)	r_L (cm)	r_R (cm)	F_L (N)	F_R (N)	$F_{L,spring}$ (N)	$F_{R,spring}$ (N)	% diff (left)	% diff (right)
100	50	50	0.5	0.5	0.45	0.45	9.9	9.9
100	30	70	0.68	0.29	0.65	0.3	4.4	3.4
100	10	90	0.9	0.1	0.85	0.1	5.5	0
200	50	50	0.98	0.98	1	1	0	0
200	30	70	1.38	0.59	1.35	0.55	2.1	7.2
200	10	90	1.8	0.2	1.85	0.2	2.7	0

Applications and Summary

All bridges are under some amount of stress, from both their own weight and the weight of the loads moving across. Suspension bridges, like the Golden Gate, are a complex system of objects under very heavy forces and in equilibrium. The cables that hold the bridge up are elastic, and their elasticity was considered when the structural engineers designed the bridge. Similarly, skyscrapers have a complex system of steel beams under tremendous forces, which altogether compose a rigid system in static equilibrium. Elasticity plays a role in the materials used to construct buildings, as they need to be able to withstand a certain amount of flexing, especially in areas where earthquakes are prevalent. The cranes used to construct these structures are also in equilibrium, with a complex system of cables and pulleys to lift and lower the construction materials.

In this study, the equilibrium of a system composed of multiple components under various forces was observed. The effects of the elastic components were also observed using spring scales of known spring constants. The forces exerted upon the springs were computed using the two conditions necessary for equilibrium: the sum of the forces and the sum of the torques are zero.